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# Interpolation by Functions in Mixed-Norm Spaces of Analytic Functions

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## 1. INTRODUCTION

Theorem 2.1.1 of Amar [1], generalized by Rochberg ([4]), says that every separated sequence in the open unit disc $D$ is the union of finitely many interpolation sequences for $A^{\beta}_{ss}$, the Bergman space of functions $f$ which are analytic in $D$ and for which

$$\int_0^1 \int_0^{2\pi} |f(\rho e^{i\theta})|^s (1-\rho)^{s\beta-1}\, d\rho\, d\theta < \infty \qquad (0 < s, \beta < \infty).$$

The purpose of this note is to prove an analogous result with $A^{\beta}_{ss}$ replaced by $A^{\beta}_{sr}$ $(0 < s, r \leqslant \infty, 0 < \beta < \infty)$, the mixed-norm space of analytic functions in $D$ (defined below). Our proof will be performed (for technical reasons) in the upper half-plane $U$ instead of the disc.

If $s, r, \beta > 0$, a function $f$ analytic in $U$, is said to belong to the space $A^{\beta}_{sr}$ if

$$\|f\|^r_{sr} = \int_0^{\infty} y^{r\beta-1} M_s(y, f)^r\, dy < \infty,$$

where

$$M_s(y, f) = \left( \int_{-\infty}^{\infty} |f(x+iy)|^s\, dx \right)^{1/s},$$

with the usual understanding if $s$ or $r = \infty$.

Let $l_{sr}$ denote the mixed-norm space of all double sequences $a = \{a_{mk}\}$, $m, k \in Z$, for which

$$\|a\|^r_{sr} = \sum_m \left( \sum_k |a_{mk}|^s \right)^{r/s} < \infty \qquad (0 < s, r < \infty).$$



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In the case where $s$ or $r$ is infinite, replace the corresponding sum by a supremum.

A sequence $\{z_{nl}\}$, $n, l \in Z$ in $U$ is said to be separated if there exists a $\delta > 0$ such that if $(n, l) \neq (m, k)$ then

$$d(z_{nl}, z_{mk}) = \left| \frac{z_{nl} - z_{mk}}{z_{nl} - \bar{z}_{mk}} \right| \geqslant \delta.$$

A sequence $\{z_{nl}\}$ in $U$ is called interpolation sequence for $A^{\beta}_{sr}$ if whenever $\{a_{nl}\} \in l_{sr}$, then there exists $f \in A^{\beta}_{sr}$ satisfying

$$f(z_{nl})(\operatorname{Im} z_{nl})^{\beta + (1/s)} = a_{nl}$$

Now we state our main result in the form of a theorem.

THEOREM. *Suppose $\{z_{nl}\}$ is a separated sequence in $U$. Then it can be split into a finite number of interpolation sequences for $A^{\beta}_{sr}$.*

## 2

Let $\mu$ and $\nu$ be positive integers and $n_0$ a nonnegative integer. Our main tool is the following theorem.

THEOREM 2.1. *Suppose $\{z_{n_0+n\nu,l} = x_{n_0+n\nu,l} + iy_{n_0+n\nu,l}\}$, $n, l \in Z$, is a sequence in $U$ which satisfies the following conditions*

$$\begin{aligned} &\text{(i)} \quad 2^{n_0+(n+1)\nu-1} \leqslant y_{n_0+n\nu,l} < 2^{n_0+(n+1)\nu} && \textit{for all} \quad l \in Z, \\ &\text{(ii)} \quad |x_{n_0+n\nu,l_1} - x_{n_0+n\nu,l_2}| \geqslant \mu 2^{n_0+(n+1)\nu-1} && \textit{if} \quad l_1 \neq l_2. \end{aligned} \tag{2.1}$$

*If $\mu$ and $\nu$ are large enough then $T^{\beta}_{sr}$ defined by*

$$T^{\beta}_{sr}(f) = \{f(z_{n_0+n\nu,l})\, y^{\beta+(1/s)}_{n_0+n\nu,l}\}, \qquad f \in A^{\beta}_{sr},$$

*is a continuous map of $A^{\beta}_{sr}$ onto $l_{sr}$. In fact, there is a continuous linear map $V$ of $l_{sr}$ into $A^{\beta}_{sr}$ so that $T^{\beta}_{sr} V = I_{l_{sr}}$.*

*Proof.* For the sake of simpicity we will write the proof for the cases where $s, r \neq \infty$. Without loss of generality we may suppose $n_0 = 0$. In the proof of the theorem the quantity $\beta + 1/s$ will be denoted by $p$.

First, we show that $T^{\beta}_{sr}$ is a bounded operator from $A^{\beta}_{sr}$ into $l_{sr}$. By Lemma 6.3 [3] $\|f\|_{sr}$ is equivalent to $\|\{2^{mp} \sup_{z \in Q_{mk}} |f(z)|\}_{sr}$, where $Q_{mk}$ are squares with vertices $k2^m + i2^m$, $(k+1)\,2^m\ + i2^m$, $k2^m + i2^{m+1}$ and $(k+1)\,2^m\ + i2^{m+1}$. Hence,

$$\|T^\beta_{sr}(f)\|^r_{sr} = \|Tf\|^r_{sr} = \sum_n \left( \sum_l |f(z_{nv,l})|^s\, y^{ps}_{nv,l} \right)^{r/s}$$
$$\leqslant \sum_n \left( \sum_k \sup_{z \in Q_{nv+v-1,k}} |f(z)|^s\, 2^{p(n+1)vs} \right)^{r/s}$$
$$\leqslant C \sum_m \left( \sum_k \sup_{z \in Q_{mk}} |f(z)|^s\, 2^{mps} \right)^{r/s} \leqslant C\, \|f\|^r_{sr}.$$

By a similar argument we have $\|Tf\|_{sr} \leqslant C\|f\|_{sr}$, if $s$ or $r = \infty$.

We use $C$ to denote a positive constant, depending on the particular parameters $r, s, \ldots, \eta, \beta, \ldots,$ concerned in the particular problem in which it appears. It is not necessarily the same on any two occurrences.

If $\eta > \max\{\beta + 1/s, \beta + 1\}$ we define $A_{nv,l}$ in $U$ by

$$A_{nv,l}(z) = \frac{(2i)^\eta (\operatorname{Im} z_{nv,l})^{\eta - p}}{(z - \bar{z}_{nv,l})^\eta}.$$

It follows from [3] that

$$R(\{a_{nl}\}) = \sum_{n,l} a_{nl} A_{nv,l}, \qquad \{a_{nl}\} \in l_{sr},$$

is a bounded linear operator from $l_{sr}$ into $A^\beta_{sr}$. We now claim that if $\mu$, $v$ are large enough then $TR - I_{l_{sr}}$ is an operator of small norm. (If $1 < s < \infty$, we have additional hypothesis $\eta > \beta + 1 + (\beta s s')^{-1}$, where $s'$ is conjugate to $s$.)

First, we consider the case $0 < s \leqslant 1$ with subcases $r \leqslant s$, $s < r < \infty$. Let $\{b_{nl}\} = \{(TR - I_{l_{sr}})(\{a_{nl}\})\}$. Then

$$\|\{b_{nl}\}\|^r_{sr} = \sum_n \left( \sum_l \left| \sum_{(m,k) \neq (n,l)} a_{mk}\, y^p_{nv,l} A_{mv,k}(z_{nv,l}) \right|^s \right)^{r/s}$$
$$\leqslant \sum_n \left( \sum_l \sum_{(m,k) \neq (n,l)} |a_{mk}|^s\, y^{ps}_{nv,l} |A_{mv,k}(z_{nv,l})|^s \right)^{r/s}$$
$$\leqslant C \left\{ \sum_n \left( \sum_l \sum_{m \neq n} \sum_k |a_{mk}|^s \frac{y^{ps}_{nv,l}\, y^{(\eta - p)s}_{mv,k}}{|z_{nv,l} - \bar{z}_{mv,k}|^{\eta s}} \right)^{r/s} \right.$$
$$\left. + \sum_n \left( \sum_l \sum_{k \neq l} |a_{nk}|^s \frac{y^{ps}_{nv,l}\, y^{(\eta - p)s}_{nv,k}}{|z_{nv,l} - \bar{z}_{nv,k}|^{\eta s}} \right)^{r/s} \right\}. \tag{2.2}$$

The first sum on the right-hand side of (2.2) will be denoted by $S_1$, the second by $S_2$. The deal with $S_1$ we first calculate

$$\sum_l |z_{nv,l} - \bar{z}_{mv,k}|^{-\eta s}, \qquad m \neq n.$$

$$\sum_l |z_{n\nu,l} - \bar{z}_{m\nu,k}|^{-\eta s}$$

$$\leqslant 2 \sum_{t=0}^{\infty} [(2^{n\nu+\nu-1} + 2^{m\nu+\nu-1})^2 + t^2\mu^2 2^{2(n\nu+\nu-1)}]^{-ns/2}$$

$$\leqslant C2^{-(n+1)\nu\eta s} \sum_{t=0}^{\infty} [t^2\mu^2 + (1+2^{(m-n)\nu})^2]^{-\eta s/2}$$

$$\leqslant C2^{-(n+1)\nu\eta s}\left[(1+2^{(m-n)\nu})^{-\eta s} + \int_0^\infty [\mu^2x^2 + (1+2^{(m-n)\nu})^2]^{-\eta s/2}\,dx\right]. \tag{2.3}$$

To get a useful estimate we need the following estimate of the integral on the right-hand side of (2.3).

$$\int_0^\infty [\mu^2x^2 + (1+2^{(m-n)\nu})^2]^{-\eta s/2}\,dx$$

$$= (1+2^{(m-n)\nu})^{-\eta s} \int_0^\infty \left[1 + \left(\frac{\mu x}{1+2^{(m-n)\nu}}\right)^2\right]^{-\eta s/2} dx$$

$$= \mu^{-1}(1+2^{(m-n)\nu})^{1-\eta s} \int_0^\infty (1+x^2)^{-\eta s/2}\,dx. \tag{2.4}$$

The last integral is convergent because $\eta s > 1$. Hence, if $m \neq n$ then from (2.3) and (2.4) we see that

$$\sum_l |z_{n\nu,l} - \bar{z}_{m\nu,k}|^{-\eta s} \leqslant C\mu^{-1}2^{-(n+1)\nu\eta s}(1+2^{(m-n)\nu})^{1-\eta s}. \tag{2.5}$$

Using (2.5) we find that

$$S_1 \leqslant C\sum_n \left(\sum_{m\neq n}\sum_k |a_{mk}|^s 2^{(n+1)\nu ps} 2^{(m+1)\nu(\eta-p)s} \sum_l |z_{n\nu,l} - \bar{z}_{m\nu,k}|^{-\eta s}\right)^{r/s}$$

$$\leqslant C\mu^{-r/s}\sum_n \left(\sum_{m\neq n} \frac{2^{(m+1)\nu(\eta-p)s}2^{(n+1)\nu(p-\eta)s}}{(1+2^{(m-n)\nu})^{\eta s-1}} \sum_k |a_{mk}|^s\right)^{r/s}. \tag{2.6}$$

If $r \leqslant s$, then we have

$$S_1 \leqslant C\mu^{-r/s}\sum_n\sum_{m\neq n} \frac{2^{(m+1)\nu(\eta-p)r}2^{(n+1)\nu(p-\eta)r}}{(1+2^{(m-n)\nu})^{\eta r-(r/s)}}\left(\sum_k |a_{mk}|^s\right)^{r/s}$$

$$= C\mu^{-r/s}\sum_m\left(\sum_k |a_{mk}|^s\right)^{r/s}\sum_{n\neq m} \frac{2^{(m+1)\nu(\eta-p)r}2^{(n+1)\nu(p-\eta)r}}{(1+2^{(m-n)\nu})^{\eta r-(r/s)}}. \tag{2.7}$$

Observe that

$$\sum_{n \neq m} \frac{2^{(m+1)v(\eta-p)r} 2^{(n+1)v(p-\eta)r}}{(1+2^{(m-n)v})^{\eta r-(r/s)}}$$

$$\leqslant \sum_{t=1}^{\infty} 2^{-tv(\eta-p)r} + \sum_{t=1}^{\infty} 2^{-\beta r v t}$$

$$= (2^{(\eta-p)rv}-1)^{-1} + (2^{\beta r v}-1)^{-1}. \tag{2.8}$$

From (2.7) and (2.8) it follows that

$$S_1 \leqslant C_1(\mu, v) \cdot \|\{a_{mk}\}\|_{sr}^{r}, \tag{2.9}$$

where $C_1(\mu, v) \to 0$, if $\mu, v \to \infty$.

We now turn to the estimate of the sum $S_2$:

$$S_2 = C \sum_n \left( \sum_l \sum_{k \neq l} |a_{nk}|^s \frac{2^{(n+1)vps} 2^{(n+1)v(\eta-p)s}}{|z_{nv,l} - \bar{z}_{nv,k}|^{\eta s}} \right)^{r/s}$$

$$= C \sum_n \left( \sum_k |a_{nk}|^s 2^{(n+1)v\eta s} \sum_{l \neq k} |z_{nv,l} - \bar{z}_{nv,k}|^{-\eta s} \right)^{r/s} \tag{2.10}$$

$$\sum_{l \neq k} |z_{nv,l} - \bar{z}_{nv,k}|^{-\eta s} \leqslant 2 \sum_{t=1}^{\infty} [(2^{nv+v-1})^2 + \mu^2 t^2 2^{2(\eta v+v-1)}]^{-\eta s/2}$$

$$\leqslant C 2^{-(n+1)v\eta s} \left( \sum_{t=1}^{\infty} t^{-\eta s} \right) \cdot \mu^{-\eta s}. \tag{2.11}$$

Since $\eta s > 1$, we see from (2.11) and (2.10) that

$$S_2 \leqslant C_2(\mu) \|\{a_{nk}\}\|_{sr}^{r}, \tag{2.12}$$

where $C_2(\mu) \to 0$, if $\mu \to \infty$. Note that (2.12) holds as long as $0 < s \leqslant 1$, $0 < r < \infty$.

If $s \leqslant r < \infty$, we start from (2.6) and obtain

$$S_1 \leqslant C\mu^{-r/s} \left\{ \sum_n \left( \sum_{t=1}^{\infty} 2^{-tv\beta s} \sum_k |a_{n+t,k}|^s \right)^{r/s} \right.$$

$$\left. + \sum_n \left( \sum_{t=1}^{\infty} 2^{-tv(\eta-p)s} \sum_k |a_{n-t,k}|^s \right)^{r/s} \right\}$$

$$
\begin{aligned}
&\leqslant C\mu^{-r/s}\left\{\left(\sum_{t=1}^{\infty} 2^{-tv\beta s}\left(\sum_n\left(\sum_k |a_{n+t,k}|^s\right)^{r/s}\right)^{s/r}\right)^{r/s}\right.\\
&\quad\left.+\left(\sum_{t=1}^{\infty} 2^{-tv(\eta-p)s}\left(\sum_n\left(\sum_k |a_{n-t,k}|^s\right)^{r/s}\right)^{s/r}\right)^{r/s}\right\}\\
&\leqslant C\mu^{-r/s}\|\{a_{nl}\}\|_{sr}^r\,((2^{v\beta s}-1)^{-r/s}+(2^{v(\eta-p)s}-1)^{-r/s})\\
&= C_3(\mu, v)\|\{a_{nl}\}\|_{sr}^r,
\end{aligned}
\tag{2.13}
$$

where $C_3(\mu, v)\to 0$, if $\mu, v\to\infty$.

From (2.2), (2.9), (2.12), and (2.13) we conclude that if $\mu$ and $v$ are large then $TR - I_{l_{sr}}$ has a small norm. $V = R(TR)^{-1}$ is the required operator. The proof is now complete in the case where $0 < s \leqslant 1$, $0 < r < \infty$.

Assume now $1 < s < \infty$ and $\eta > \beta + 1 + 1/\beta s s'$. Some simple observations show that it is possible to select two positive $\theta$ and $\tau$ such that

$$
\begin{aligned}
&\text{(i)}\quad \theta+\tau = 1\\
&\text{(ii)}\quad \eta\tau s' > 1\\
&\text{(iii)}\quad (\eta-p)\,\tau s' > 1\\
&\text{(iv)}\quad \eta\theta s > 1\\
&\text{(v)}\quad \theta p s > 1
\end{aligned}
\tag{2.14}
$$

It is clear that (iii) and (v) imply (ii) and (iv), respectively. Using $\eta = \beta + 1 + \varepsilon$, $\varepsilon > (\beta s s')^{-1}$, we may rewrite the conditions as follows:

(iii) $(\eta - p)(1-\theta) > 1/s'$ if and only if $\theta(1+\varepsilon s') < \varepsilon s'$

(v) $\theta p s > 1$ if and only if $\theta(1+\beta s) > 1$.

The condition $\varepsilon\beta s s' > 1$ is equivalent to $1 + \varepsilon s' < \varepsilon s'(1+\beta s)$. Therefore, the interval defining possible choices for $\theta$ is nonempty. Choose $\theta$ and $\tau$ so that (2.14) is satisfied. Then

$$
\begin{aligned}
\|\{b_{nl}\}\|_{sr}^r \leqslant C&\left\{\sum_n\left[\sum_{nl}\left(\sum_{m\neq n}\sum_k |a_{mk}|^s \frac{2^{(m+1)v\theta s(\eta-p)}2^{(n+1)vps}}{|z_{nv,l}-\bar z_{mv,k}|^{\eta\theta s}}\right)\right.\right.\\
&\left.\times\left(\sum_{m\neq n}\sum_k \frac{2^{(m+1)v(\eta-p)\tau s'}}{|z_{nv,l}-\bar z_{mv,k}|^{\eta\tau s'}}\right)^{s/s'}\right]^{r/s}\\
&+\sum_n\left[\sum_l\left(\sum_{k\neq l}|a_{nk}|^s\frac{2^{(n+1)v\eta s}}{|z_{nv,l}-\bar z_{nv,k}|^{\eta\theta s}}\right)\right.\\
&\left.\left.\times\left(\sum_{k\neq l}\frac{1}{|z_{nv,l}-\bar z_{mv,k}|^{\eta\tau s'}}\right)^{s/s'}\right]^{r/s}\right\}.
\end{aligned}
$$

If $n \neq m$, then, as in (2.5),

$$\sum_k |z_{n\nu,l} - \bar{z}_{m\nu,k}|^{-\eta\tau s'} = \sum_k |z_{m\nu,k} - \bar{z}_{n\nu,l}|^{-\eta\tau s'}$$
$$\leqslant C\mu^{-1} 2^{-(m+1)\nu\eta\tau s'} (1 + 2^{(n-m)\nu})^{1-\eta\tau s'}$$

(Here we have used the fact that $\eta\tau s' > 1$). Now,

$$I_1 = \left( \sum_{m \neq n} \sum_k \frac{2^{(m+1)\nu(\eta - p)\tau s'}}{|z_{n\nu,l} - \bar{z}_{m\nu,k}|^{\eta\tau s'}} \right)^{s/s'}$$
$$\leqslant C\mu^{-s/s'} \left( \sum_{m \neq n} \frac{2^{-(m+1)\nu p\tau s'}}{(1 + 2^{(n-m)\nu})^{\eta\tau s' - 1}} \right)^{s/s'}$$
$$\leqslant C\mu^{-s/s'} 2^{-(n+1)\nu p\tau s} \left( \sum_{t=1}^{\infty} 2^{-t\nu p\tau s'} + \sum_{t=1}^{\infty} 2^{-t\nu[(\eta - p)\tau s' - 1]} \right)^{s/s'}.$$

Since $(\eta - p)\,\tau s' > 1$, we have

$$I_1 \leqslant \sigma_1(\mu, \nu)\, 2^{-(n+1)\nu p\tau s}, \qquad \text{where } \sigma_1(\mu, \nu) \to 0, \text{ as } \mu, \nu \to \infty.$$

Thus, the first sum $\sigma_1$ on the right-hand side of (2.15) is at most

$$\sigma_1 \leqslant [\sigma_1(\mu, \nu)]^{r/s} \sum_n \left( \sum_l \sum_{m \neq n} \sum_k |a_{mk}|^s \frac{2^{(m+1)\nu(\eta - p)\theta s} 2^{(n+1)\nu p\theta s}}{|z_{n\nu,l} - \bar{z}_{m\nu,k}|^{\eta\theta s}} \right)^{r/s}$$
$$= [\sigma_1(\mu, \nu)]^{r/s}$$
$$\times \sum_n \left( \sum_{m \neq n} \sum_k |a_{mk}|^s\, 2^{(m+1)\nu(\eta - p)\theta s} 2^{(n+1)\nu p\theta s} \sum_l |z_{n\nu,l} - \bar{z}_{m\nu,k}|^{-\eta\theta s} \right)^{r/s}. \tag{2.16}$$

Since $\eta\theta s > 1$, we have for $m \neq n$,

$$\sum_l |z_{n\nu,l} - \bar{z}_{m\nu,k}|^{-\eta\theta s} \leqslant C 2^{-(n+1)\nu\eta\theta s} (1 + 2^{(m-n)\nu})^{1-\eta\theta s}. \tag{2.17}$$

Now we substitute (2.17) into (2.16) to get

$$\sigma_1 \leqslant C[\sigma_1(\mu, \nu]^{r/s} \sum_n \left( \sum_{m \neq n} \sum_k |a_{mk}|^s \frac{2^{(m+1)\nu(\eta - p)\theta s} 2^{(n+1)\nu(p - \eta)\theta s}}{(1 + 2^{(m-n)\nu})^{\eta\theta s - 1}} \right)^{r/s}. \tag{2.18}$$

If $r \leqslant s$, we proceed as in (2.7) to get

$$\sigma_1 \leqslant \sigma_2(\mu, \nu) \, \| \{a_{mk}\} \|_{sr}^r, \qquad \text{where} \quad \sigma_2(\mu, \nu) \to 0, \quad \text{if} \quad \mu, \nu \to \infty. \tag{2.19}$$

(Here we needed the restriction $\theta ps > 1$.) If $s \leqslant r < \infty$, and if $\theta ps > 1$ then

$$\sigma_1 \leqslant \sigma_3(\mu, \nu) \| \{a_{mk}\} \|_{sr}^r, \qquad \text{where} \quad \sigma_3(\mu, \nu) \to 0, \text{ if } \mu, \nu \to \infty. \tag{2.20}$$

As in (2.11), if $\eta \tau s' > 1$, then

$$\left( \sum_{k \neq l} |z_{n\nu,l} - \bar{z}_{n\nu,k}|^{-\eta \tau s'} \right)^{s/s'} = \left( \sum_{k \neq l} |z_{n\nu,k} - \bar{z}_{n\nu,l}|^{-\eta \tau s'} \right)^{s/s'}$$
$$\leqslant C \mu^{-\eta \tau s} 2^{-(n+1)\nu \tau s \eta}.$$

Using this we find that the second sum $\sigma_2$ on the right-hand side of (2.15) is at most

$$\sigma_2 \leqslant C \mu^{-\eta \tau r} \sum_n \left( \sum_l \sum_{k \neq l} |a_{nk}|^s \frac{2^{(n+1)\nu \eta \theta s}}{|z_{n\nu,l} - \bar{z}_{n\nu,k}|^{\eta \theta s}} \right)^{r/s}$$
$$= C \mu^{-\eta \tau r} \sum_n \left( \sum_k |a_{nk}|^s \, 2^{(n+1)\nu \eta \theta s} \sum_{l \neq k} |z_{n\nu,l} - \bar{z}_{n\nu,k}|^{-\eta \theta s} \right)^{r/s}$$
$$\leqslant \sigma_4(\mu) \| \{a_{nl}\} \|_{sr}^r, \qquad \text{where} \quad \sigma_4(\mu) \to 0, \text{ as } \mu \to \infty. \tag{2.21}$$

(Here we needed $\eta \theta s > 1$.)

From (2.15), (2.19), (2.20), and (2.21) we see that if $\mu$ and $\nu$ are large enough then $S = TR - I_{l_{sr}}$ has small norm. $V = R(I_{l_{sr}} + S)^{-1}$ is the required operator. This completes the proof.

## 3

Our main theorem is now an easy consequence of Theorem 2.1.

*Proof of the Theorem.* We show that $X = \{z_{nl}\}$ can be split into a finite number of sequences $X_i$ which satisfy condition (2.1). Since $X$ is separated, there exists a positive integer $M$ such that each square $Q_{mk}$ of the partition of $U$, described in Section 2, contains at most $M$ points of the sequence $X$. Hence, we can devide $X$ into $M$ sequences $x_i$ such that each of them has at most one point in any square $Q_{mk}$.

Consider now $\mu + 1$ sets of indices $\Lambda_k = \{(n, l): n \in Z, l = k (\bmod \mu)\}$, $k = 1, 2, ..., \mu + 1$. Now split each of the sequences $x_i$ into a $\mu + 1$ sequences $x_{ij}$, $j = 1, 2, ..., \mu + 1$, defined by

$$z_{nl} \in x_{ij} \text{ if and only if } z_{nl} \in x_i \cap Q_{mk}, \qquad (m, k) \in \Lambda_j.$$

Let $L_k = \{(n, l): n = k (\bmod \nu), l \in Z\}$, $k = 1, 2, ..., \nu$. Finally, we split each $x_{ij}$ into a $\nu$ sequences $x_{ijk}$, $k = 1, 2, ..., \nu$, so that $z_{nl} \in x_{ijk}$ if and only if

$z_{nl} \in x_{ij} \cap Q_{mu}$, $(m, u) \in I_k$. To finish the proof, we enumerate the sequences $x_{ijk}$ so that (2.1) is satisfied.

## 4. Remarks

1. The following lemma is an analog of the Lemma 2.1 [4].

LEMMA 4.1. *There is a constant* $C = C(s, r, \beta)$ *so that for all* $z_1, z_2 \in U$ *with* $d(z_1, z_2) < 1$

$$|(\operatorname{Im} z_1)^{\beta + 1/s} f(z_1) - (\operatorname{Im} z_2)^{\beta + 1/s} f(z_2)| = Cd(z_1, z_2)\, \|f\|_{sr}.$$

As a corollary we obtain that if $\{z_{nl}\}$ is an interpolating sequence for $A^{\beta}_{sr}$ then it is separated.

2. In Section 2 we have proved that if $\{z_{n_0 + n\nu, l}\}$ satisfies (2.1) then $T^{\beta}_{sr}$ is bounded operator from $A^{\beta}_{sr}$ into $l_{sr}$. A similar argument establishes the following:

PROPOSITION 4.2. *Suppose* $\{z_{nl}\}$ *is a separated sequence in U. Then* $F^{\beta}_{sr}$ *defined by*

$$F^{\beta}_{sr}(f) = \{f(z_{nl})(\operatorname{Im} z_{nl})^{\beta + 1/s}\}, \qquad f \in A^{\beta}_{sr},$$

*is a bounded operator from* $A^{\beta}_{sr}$ *into* $l_{sr}$.

Conversely, if $F^{\beta}_{sr}$ is bounded operator from $A^{\beta}_{sr}$ into $l_{sr}$ then there exists $C > 0$ such that

$$\|F^{\beta}_{sr}(f)\|_{sr} \leqslant C \|f\|_{sr}, \qquad \text{for all} \quad f \in A^{\beta}_{sr}.$$

Especially,

$$\|F^{\beta}_{sr}(A_{nl})\| \leqslant C \|A_{nl}\|_{sr},$$

where

$$A_{nl}(z) = \frac{(\operatorname{Im} z_{nl})^{\eta - (\beta + 1/s)}}{(z - \bar{z}_{nl})^{\eta}}, \qquad \eta > \max\left(\beta + 1, \beta + \frac{1}{s}\right).$$

Functions $A_{nl}$ form a bounded subset of $A^{\beta}_{sr}$ [3]. Therefore

$$\sup_{n,l} \sum_{m} \left( \sum_{k} \left[ \frac{(\operatorname{Im} z_{mk})^{\beta + 1/s} (\operatorname{Im} z_{nl})^{\eta - (\beta + 1/s)}}{|z_{mk} - \bar{z}_{nl}|^{\eta}} \right]^{s} \right)^{r/s} \leqslant C < \infty. \tag{4.1}$$

The following proposition now follows from (4.1) and the theorem [6, p. 331]. We omit the details.

PROPOSITION 4.3. *Suppose* $\{z_{nl}\}$ *is a sequence in* $U$. *If* $F_{sr}^{\beta}$ *is a bounded operator from* $A_{sr}^{\beta}$ *into* $l_{sr}$ *then* $\{z_{nl}\}$ *is a finite union of separated sequences.*

## REFERENCES


1. E. AMAR, Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $C^n$, *Canad. J. Math.* **30** (1978), 711–737.
2. R. R. COIFMAN AND R. ROCHBERG, Representation theorems for holomorphic and harmonic functions in $L^p$, *Astérisque* **77** (1980), 11–66.
3. F. RICCI AND M. TAIBLESON, Boundary values of harmonic functions in mixed-norm spaces and their atomic structure, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **10**, No. 1 (1983), 1–54.
4. R. ROCHBERG, Interpolation by functions in Bergman spaces, *Michigan Math. J.* **29** (1982), 229–236.
5. S. A. VINOGRADOV, On free interpolation in Bergman spaces, *LOMI* **113** (1981), 207–211.
6. T. M. WOLNIEWITZ, Inclusion operators in Bergman spaces on bounded symmetric domains in $C^n$, *Studia Math.* **78** (1984), 329–337.